PROPAGATION IN INHOMOGENEOUS SLAB WAVEGUIDES

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Abstract

A new asymptotic method, based on evanescent wave tracking, is presented for the determination of guided modes on slab waveguides with inhomogeneous permittivity profile.

Summary

Exact solutions for the propagation characteristics of inhomogeneous film waveguides can be obtained only for a few permittivity profiles[1-3]. If the permittivity varies slowly over a length interval equal to the local wavelength, the class of solutions can be enlarged by recourse to asymptotic (WKB) methods[1]. Here we present an entirely new approach that furnishes the asymptotic expansion of the exact solution of the wave equation without the limitations imposed by the WKB procedure. The new approach is based on evanescent wave tracking whereby local evanescent plane wave fields are matched to an inhomogeneity profile in such a manner that the composite field behaves as a guided mode.

The general method has been described elsewhere[4] but its systematic application to the guided mode problem, as developed here, has not previously been reported. We assume a field solution of the form

$$u(\underset{\sim}{r}) \sim \exp[ik\psi(\underset{\sim}{r})], \quad \underset{\sim}{r} = (x, z) \tag{1}$$

where k is the wavenumber in vacuum and

$$\psi(\underset{\sim}{r}) = S(\underset{\sim}{r}) + \sum_{m=0}^{\infty} \frac{A_m(\underset{\sim}{r})}{k^{m+1}} . \tag{2}$$

When (1) and (2) are substituted into the two-dimensional (y-independent) wave equation with variable refractive index n(x),

$$\left[\nabla^2 + k^2 n^2(x)\right] u(\underset{\sim}{r}) = 0, \quad \nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial z^2, \tag{3}$$

and the coefficient of each power of k is set equal to zero, one obtain the sequence of equations

$$(\nabla S)^2 = n^2 \tag{4}$$

$$\nabla^2 S + 2\nabla S \cdot \nabla A_o = 0 \tag{5}$$

$$i2\nabla S \cdot \nabla A_m + \sum_{j=0}^{m-1} \nabla A_j \cdot \nabla A_{m-j-1} + \nabla^2 A_{m-1} = 0, \quad m = 1, 2 \ldots \tag{6}$$

The eikonal equation (4) and the lowest order transport equation (5) yield the local plane wave field

$$u_o(\underset{\sim}{r}) \sim \exp[ikS(\underset{\sim}{r})] \exp[A_o(\underset{\sim}{r})], \tag{7}$$

while the higher order transport equations in (6) provide correction terms. By assuming complex phase and amplitude functions

$$S(\underset{\sim}{r}) = R(\underset{\sim}{r}) + iI(\underset{\sim}{r}), \quad A_m(\underset{\sim}{r}) = w_m(\underset{\sim}{r}) + iv_m(\underset{\sim}{r}), \tag{8}$$

where R, I, $w_m$ and $v_m$ are real, and defining the phase paths and equiphase contours tangent to the perpendicular unit vectors $\underset{\sim}{s}_o$ and $\underset{\sim}{t}_o$, respectively,[4]

$$\underset{\sim}{s}_o = \frac{\nabla R}{\beta}, \quad \underset{\sim}{t}_o = \frac{\nabla I}{\alpha}, \text{ with } \beta = |\nabla R|, \ \alpha = |\nabla I| \tag{9}$$

one may derive the trajectory equations

$$\frac{d}{ds}(\beta \underset{\sim}{s}_o) = \nabla\beta, \quad \frac{d}{dt}(\alpha \underset{\sim}{t}_o) = \nabla\alpha, \tag{10}$$

and, from (4),

$$\beta^2 - \alpha^2 = n^2. \tag{11}$$

Here, s and t are lengths measured along the phase paths I = constant and the equiphase contours R = constant, respectively. The transport equations become

$$2\beta \frac{dw_o}{ds} - 2\alpha \frac{dv_o}{dt} + \nabla \cdot (\beta \underset{\sim}{s}_o) = 0 \tag{12}$$

$$2\beta \frac{dv_o}{ds} + 2\alpha \frac{dw_o}{dt} + \nabla \cdot (\alpha \underset{\sim}{t}_o) = 0 \tag{13}$$

and for $m \geq 1$,

$$-2\left(\alpha \frac{dw_m}{dt} + \beta \frac{dv_m}{ds}\right) + \sum_{j=0}^{m-1} (\nabla w_j \cdot \nabla w_{m-j-1} - \nabla v_j \cdot \nabla v_{m-j-1}) + \nabla^2 w_{m-1} = 0 \tag{14}$$

$$2\left(\beta \frac{dw_m}{ds} - \alpha \frac{dv_m}{dt}\right) + \sum_{j=0}^{m-1} (\nabla v_j \cdot \nabla w_{m-j-1} + \nabla w_j \cdot \nabla v_{m-j-1}) + \nabla^2 v_{m-1} = 0 \tag{15}$$

The solution of these equations is accomplished[4] by determining the phase paths and equiphase contours from (10), utilizing the trajectory grid to find $\alpha = dI/dt$ or $\beta = dR/ds$, integrating along the appropriate path to find I or R, and finally performing the integrations in (12) - (15) to find $w_m$ and $v_m$. Note that only $\alpha$ or $\beta$ need be found by integration since these parameters are related via (11).

When these equations are applied to guided mode propagation along z, the following conditions must be imposed: 1. the equiphase contours should be planes perpendicular to z (i.e., $x \equiv t$, $z \equiv s$); 2. the phase variation of u should be linear; 3. the amplitude of u should be independent of z. Hence

$$\beta = n_o = \text{constant}, \quad \alpha(x) = \left[n_o^2 - n^2(x)\right]^{1/2} \qquad (16)$$

The requirements $\partial w_o/\partial z = 0 = \partial v_o/\partial x$ (see conditions 1. and 3. ) then imply that (12) is satisfied exactly and that (13) can be integrated to yield

$$w_o = \ln \alpha^{-1/2} + n_o p \int \frac{dx}{\alpha(x)} \, , \qquad (17)$$

where $p \equiv \partial v_o/\partial z$ is a constant (condition 2). Then the lowest order field $u_o(\underset{\sim}{r})$ becomes

$$u_o(\underset{\sim}{r}) \sim [\alpha(x)]^{-1/2} \exp\left[ikn_o z - k\int \alpha(x)\, dx - ipz + n_o p \int dx/\alpha(x)\right] , \qquad (18)$$

with p chosen so that $u_o$ remains bounded for all x. Note that (18) is an exact solution of (11), (12) and (13). By applying conditions 1. - 3. to the higher order transport equations (14) and (15), one finds that these can also be integrated exactly.

To test the validity of the procedure described above, we have applied it to two profiles for which exact solutions of the wave equation are available[2,3]:

$$\text{a) } n^2(x) = n_o^2 - a^2x^2, \quad \text{b) } n^2(x) = n_o^2 \operatorname{sech}^2 bx \qquad (19)$$

In each case, we have found that (18) and the solution of the higher order transport equations furnish the asymptotic expansion (in inverse powers of k) of the exact solution, which involves Hermite polynomials for case a) and Legendre functions for case b). We have also been able to generate higher order guided modes in addition to the lowest order mode whose field has a maximum at x = 0 and monotonic decay for $|x| > 0$. The manner of derivation of these results will be described in detail.

Having confirmed the validity of the method, we have treated various permittivity profiles (in particular, polynomial variations) for which exact solutions of the wave equation are not available. The resulting modal waveforms and dispersion properties will be presented. It is also possible to treat frequency dependent refractive indexes.

The procedure in this paper holds considerable promise for enlarging the reservoir of analytical solutions for modal propagation in inhomogeneous thin film waveguides, and for aiding in the synthesis of low dispersion profiles. Even when explicit analytical solutions cannot be found, the numerical treatment of the transport equations (12) - (15) is considerably simpler than that of the wave equation.

## References


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